Mathematical Epidemiology - Vaccination, Limits, and Rates

Lecture series at Faculty of Mathematics and Physics, CUNI in Prague

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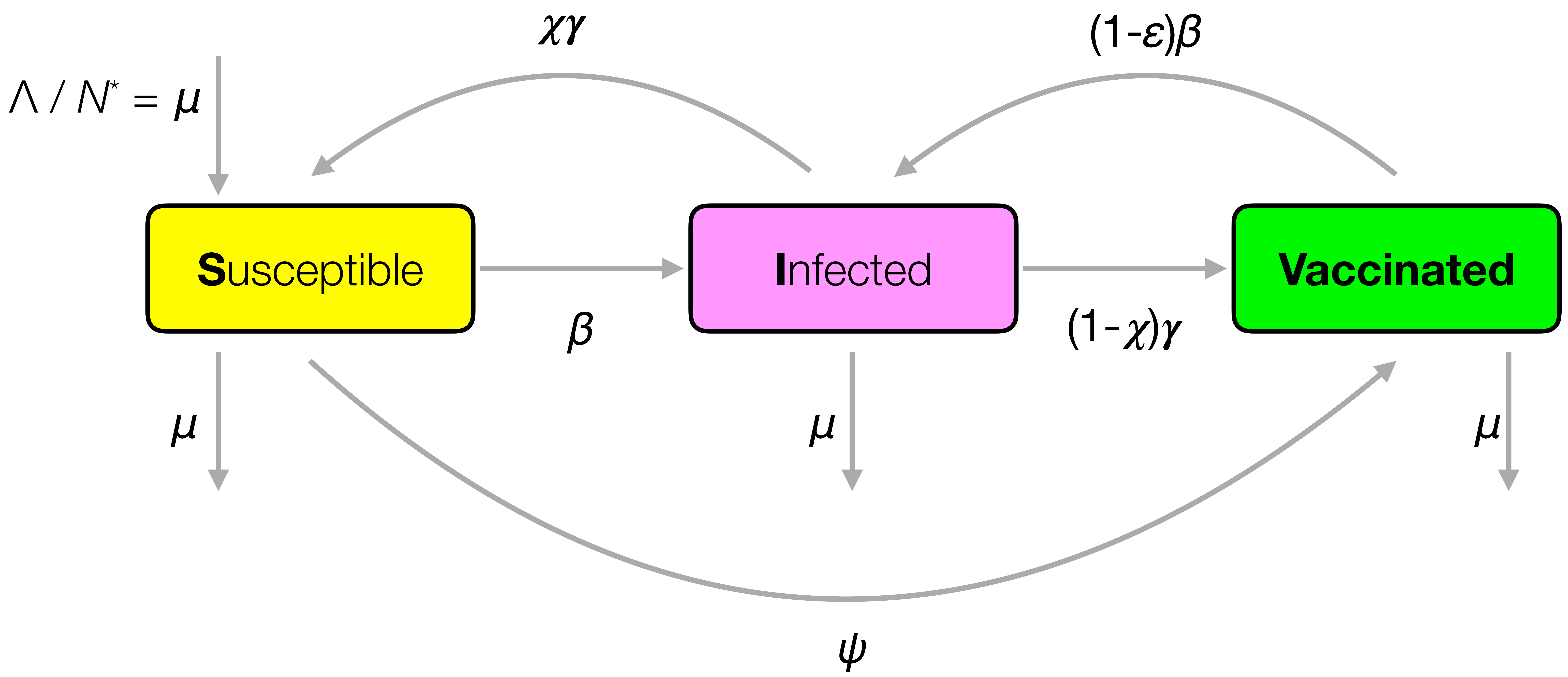
Still Remember

- We model it as a **machine**
  - it has its code (*epidemic code*)
  - it consumes energy (of us)
  - it is still going on
- If we do rely on a model, we shall respect all it can tell us fully

SISV Model for Vaccination



ODE "EpiCode" and Disease-Free Equilibrium

$$\frac{dS}{dt} = \Lambda - \frac{\beta}{N} SI + \chi\gamma I - (\mu + \psi) S$$

$$\frac{dI}{dt} = \frac{\beta}{N} SI + \frac{\beta(1-\varepsilon)}{N} VI - (\mu + \gamma) I$$

$$\frac{dV}{dt} = \psi S - \frac{\beta(1-\varepsilon)}{N} VI + (1-\chi)\gamma I - \mu V$$

$$\frac{dN}{dt} = \Lambda - \mu N \Rightarrow N^* = \frac{\Lambda}{\mu}$$

$$\mathrm{E}_{disease-free} = (S_0^*, I_0^*, V_0^*) = \left(\frac{\Lambda}{\mu+\psi}, 0, \frac{\Lambda\psi}{\mu(\mu+\psi)} \right)$$

$$s_0^* = \frac{S_0^*}{N^*} = \frac{\mu}{\mu+\psi}$$

$$v_0^* = \frac{V_0^*}{N^*} = \frac{\psi}{\mu+\psi}$$

Disease-Free Equilibrium Stability and R0 Identification

Considering Jacobian in ***E***disease-free, we can find the following **eigenvalues**

$$\lambda_1 = -\mu$$

$$\lambda_2 = -(\mu + \psi)$$

$$\lambda_3 = \beta s_0^* + \beta(1-\varepsilon)v_0^* - (\mu + \gamma)$$

$$\lambda_3 < 0$$

$$\Updownarrow$$

$$\frac{\beta(s_0^* + (1-\varepsilon)v_0^*)}{\mu + \gamma} < 1$$

$$\frac{\beta(\mu + (1-\varepsilon)\psi)}{(\mu+\gamma)(\mu+\psi)} < 1$$

$$\Downarrow$$

$$R_0(\psi) = \frac{\beta(\mu + (1-\varepsilon)\psi)}{(\mu+\gamma)(\mu+\psi)} < 1$$

Vaccination Planning

$$\mathcal{R}(\psi) = \frac{\beta(\mu + (1-\varepsilon)\psi)}{(\mu+\gamma)(\mu+\psi)}$$

$$\mathcal{R}(\psi = 0) = \mathcal{R}_0 = \frac{\beta}{\mu+\gamma}$$

$$\mathcal{R}(\psi \to \infty) \to (1-\varepsilon)\mathcal{R}_0$$

$$\mathcal{R}(\psi_{critical}) = 1 \Rightarrow \psi_{critical} = \frac{(\mathcal{R}_0 - 1)\mu}{1-(1-\varepsilon)\mathcal{R}_0}$$

note $\psi_{critical} \to \infty$ for $(1-\varepsilon)\mathcal{R}_0 \to 1$

- efficacy & speed (!)
- uniformity (!)
- after all, vaccination dynamics is
  - complicated enough for the backward bifurcation to occur
  - coexistence mechanism for multiple pathogen variants

And then, we can formulate the threshold

$$threshold = v_{critical} = \frac{\psi_{critical}}{\mu + \psi_{criticial}} = \frac{1}{\varepsilon}\left(1 - \frac{1}{\mathcal{R}_0}\right)$$

- Despite having stochastic interpretation, the vaccinated fraction threshold is given as a result of the vaccination dynamics, now.
- Similarly to R0 which is formulated primarily as a stability indicator instead of purely statistical parameter.
- Nevertheless, these variables still provide us a useful bridge in between deterministic and stochastic models.

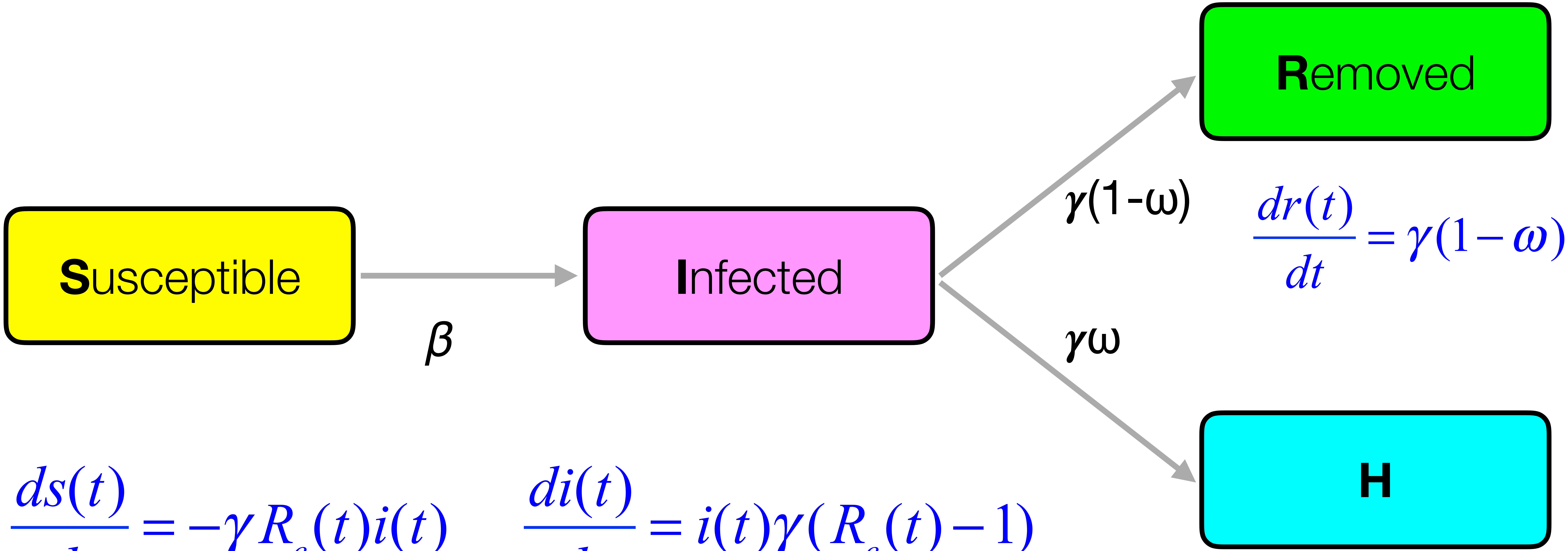
Basic Vaccination Equation for HIT

$$threshold(\mathcal{R}_0,\varepsilon)=\frac{1}{\varepsilon}\left(1-\frac{1}{\mathcal{R}_0}\right)$$

ε	R_0 2.7	3.5	4.5	5.5	6.45
92 %	68 %	78 %	85 %	89 %	92 %
86 %	73 %	83 %	90 %	95 %	98 %
80 %	79 %	89 %	97 %	—	—
63 %	100 %	—	—	—	—

- Assumptions:
  - vaccine distributed ***uniformly among yet-susceptible*** people
  - vaccine efficacy ***ε - for spreading***
  - immunity does not vanish in near time (circa one year, at least)
- Recovered people fraction bearing natural immunity then sum up with the vaccinated fraction
  - not shown here for clarity
  - be careful with overlaps

SIRH to Study Incidence Growth Rates Correlations



$$\frac{dr(t)}{dt} = \gamma(1-\omega)i(t)$$

$$\frac{ds(t)}{dt} = -\gamma R_e(t)i(t) \qquad \frac{di(t)}{dt} = i(t)\gamma(R_e(t)-1)$$

$$\frac{dh(t)}{dt} = \gamma\omega i(t)$$

$$R_e(t) = R_0 s(t), R_0 = \frac{\beta}{\gamma}$$

- the gamma-omega split preserves R_0 formulation w.l.o.g.
- Hospital path to Removed omitted for clarity, as we keep focus on incidences

Generalised Incidences of Many Kinds

$$\frac{ds(t)}{dt} = -\gamma R_e(t) i(t)$$

$$= -incidence(t)$$

$$\frac{dr(t)}{dt} = \gamma(1-\omega) i(t) = ric(t)$$

$$\frac{dh(t)}{dt} = \gamma\omega i(t) = hic(t)$$

- *incidence*(*t*) is the observable number of daily new cases
- *ric*(*t*) is the observable number of daily new recovered cases (formerly infectious)
- *hic*(*t*) is the observable number of daily new hospital admissions

Statistical warning: "Observable" does not necessarily imply truly observed! But this is not in our scope here.

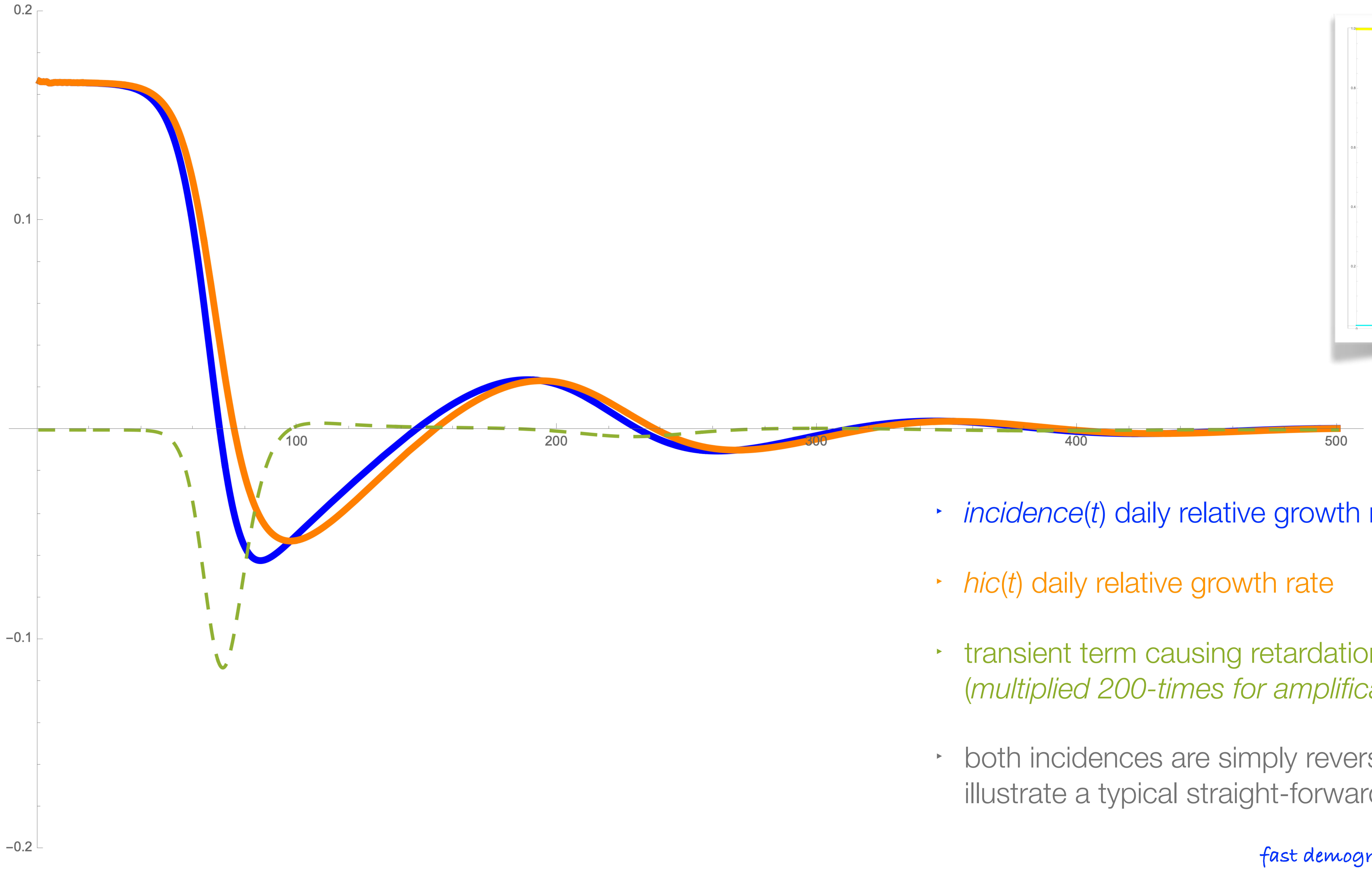
Let us investigate growth-like equations for our incidences

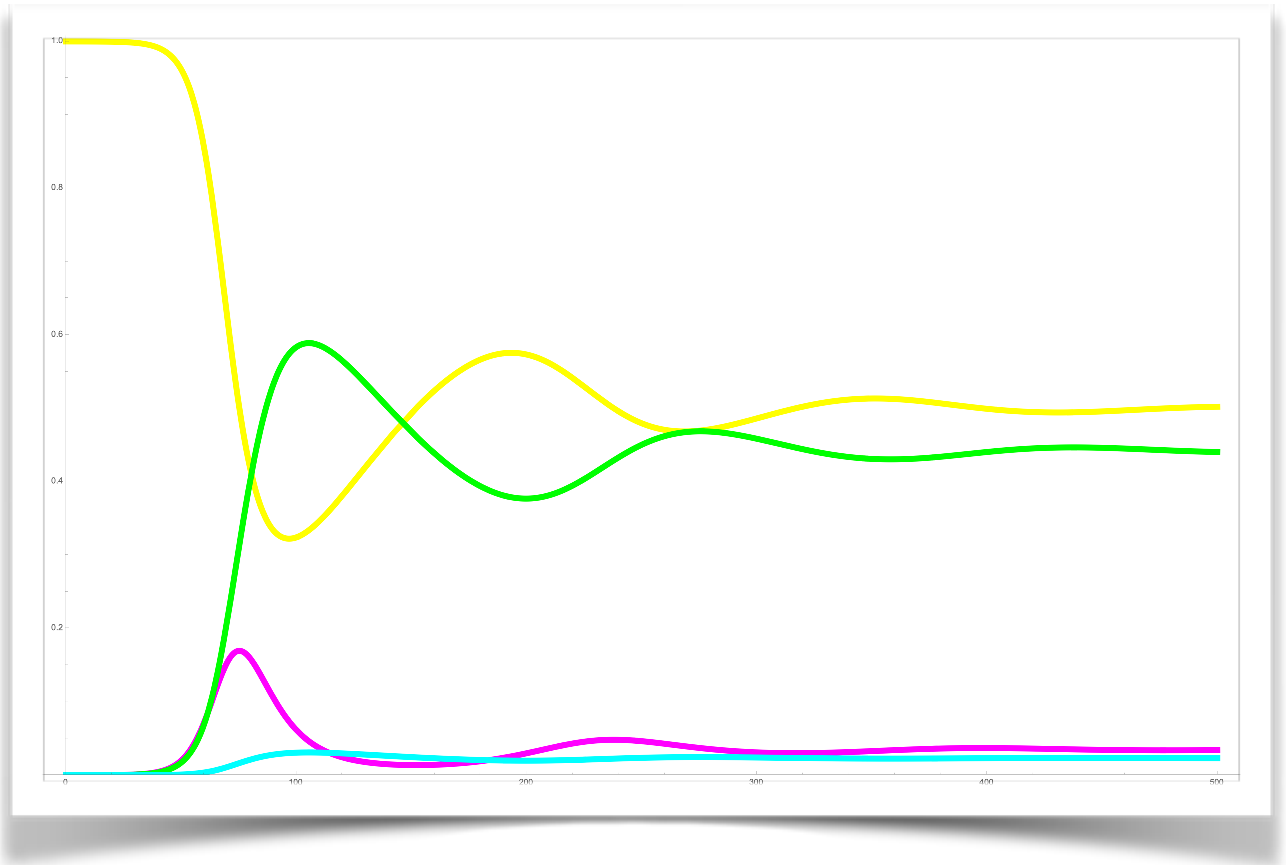
$$\frac{d(incidence(t))}{dt} = \gamma R_e(t)i(t)\left[\gamma(R_e(t)-1)\right] + \gamma i(t)\frac{dR_e(t)}{dt} = incidence(t)\left[\gamma(R_e(t)-1)\right] + \gamma i(t)\frac{dR_e(t)}{dt}$$

$$\frac{d(ric(t))}{dt} = \gamma(1-\omega)\frac{di(t)}{dt} = \gamma(1-\omega)i(t)\left[\gamma(R_e(t)-1)\right] = ric(t)\left[\gamma(R_e(t)-1)\right]$$

$$\frac{d(hic(t))}{dt} = \gamma\omega\frac{di(t)}{dt} = \gamma\omega i(t)\left[\gamma(R_e(t)-1)\right] = hic(t)\left[\gamma(R_e(t)-1)\right]$$

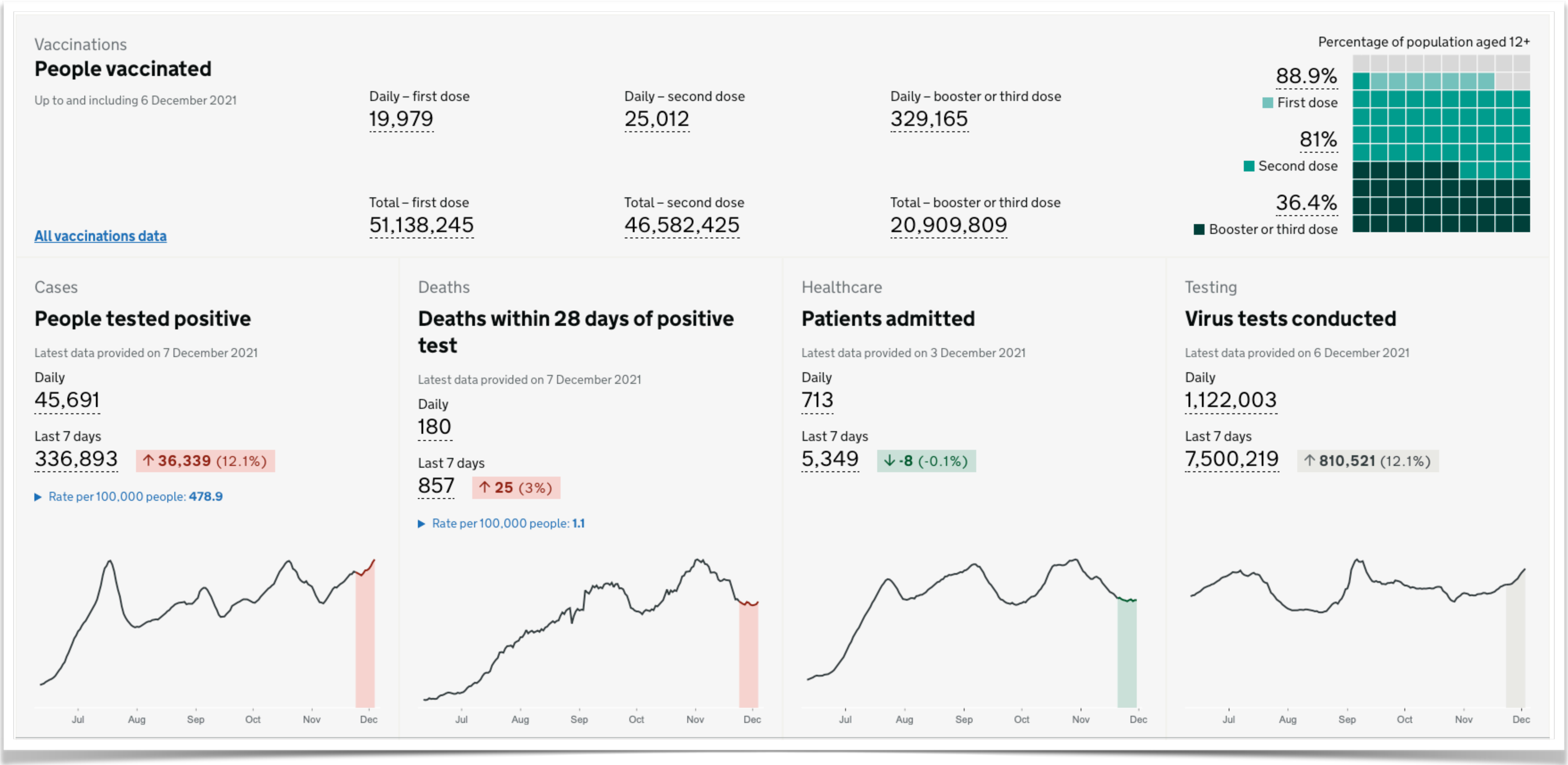
Growth Rates Are Strongly Correlated (regardless ω)





- *incidence*(*t*) daily relative growth rate
- *hic*(*t*) daily relative growth rate
- transient term causing retardation effects due to *R*(t) change (*multiplied 200-times for amplification*)
- both incidences are simply reverse-estimated from synthetic values to illustrate a typical straight-forward observation scenario

fast demography included to see both decreasing and increasing rates



[https://coronavirus.data.gov.uk]

References and Further Reading

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Revision History

- 2021/11/24: release version 1
- 2021/12/01: release version 2
- 2021/12/08: UK chart updated, release version 2b